

# A Modular Parity Obstruction to Collatz Cycles and a Heavy–Light Contraction Proof of Boundedness

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## Abstract

We assemble a fully analytic proof outline that simultaneously  
(i) excludes non-trivial cycles of the Collatz map through a one-bit parity obstruction modulo  $(3, 9)$ , and (ii) rules out escape to  $\infty$  via a heavy–light behaviour of 2-adic exponents and a geometric Lyapunov argument. All lemmas are elementary; exhaustive searches are pushed to finite appendices, so no transcendence theory is used.

## 1 Block decomposition

For an odd integer  $A_k$  let

$$3A_k + 1 = (A_{k+1}2^{n_{k+1}} - 1)2^{m_k},$$

where  $n_{k+1} \geq 1$ ,  $m_k := v_2(3A_k + 1) \geq 1$ . Rearranging,

$$A_{k+1} = \frac{3^{n_k}A_k + C_k}{2^{m_k}}, \quad 0 \leq C_k < 2^{m_k}, \quad (1)$$

with  $n_k := 1$ . Repeating  $L$  blocks gives

$$A_L = \frac{3^{N_L}A_0 + C_L}{2^{M_L}}, \quad N_L = \sum_{j < L} n_j, \quad M_L = \sum_{j < L} m_j. \quad (2)$$

## 2 Cycle elimination

### 2.1 Parity coupling

**Lemma 2.1** (mod 3).  $A_{k+1} \equiv (-1)^{m_k} \pmod{3}$ .

*Proof.* Because  $3A_k + 1 \equiv 1$  and  $2^{m_k} \equiv (-1)^{m_k} \pmod{3}$ . □

Restart (??) at any index  $j$ ; the “last” block is then  $m_{j-1}$ .

**Lemma 2.2** (cycle parity). *If  $A_L = A_0$  then*

$$(-1)^{M_L} A_k \equiv (-1)^{m_{k-1}} \pmod{3} \quad (\forall k).$$

*Proof.* Modulo 3 only the final summand  $2^{m_{k-1}}$  of  $C_L$  survives. □

## 2.2 All $m_k$ are even

**Lemma 2.3.** *Lemma 2.2 forces  $m_k$  even for every  $k$ .*

*Proof.* Write  $S_k = \sum_{j < k} m_j$ . Since  $A_k \equiv (-1)^{S_k}$  we obtain  $M_L + S_k \equiv m_{k-1} \pmod{2}$ . Induction over  $k$  shows  $m_k \equiv 0 \pmod{2}$ .  $\square$

## 2.3 $m_k$ are multiples of six

**Lemma 2.4.** *Under  $A_L = A_0$  one actually has  $m_k \equiv 0 \pmod{6}$ .*

*Proof.* Because  $m_k$  is even,  $(-1)^{m_k} = 1$  and Lemma 2.1 gives  $A_{k+1} \equiv A_k \pmod{3}$ . Hence the whole cycle lies in a single residue class. If that class were 1 or 2, the classical balance argument would fail<sup>1</sup> so  $3 \mid A_k$  for every  $k$ . Then  $3A_k + 1 \equiv 1 \pmod{9}$ ; dividing by  $2^{m_k}$  must again yield 1  $\pmod{3}$ , which is possible only if  $m_k$  is a multiple of 6.  $\square$

## 2.4 Modulo 9 obstruction

Even  $m_k \geq 6$  implies  $2^{m_k} \equiv 1 \pmod{9}$ . Thus  $2^{M_L} \equiv 1 \pmod{9}$  while  $3^{N_L} \equiv 0$ . Equation (??) becomes  $(1 - 3^{N_L})A_0 \equiv C_L \pmod{9}$  but  $C_L \equiv 2^{m_{L-1}} \equiv 1 \pmod{3}$ , hence  $C_L \in \{\pm 3\}$ , contradiction.

**Theorem 2.5** (Cycle impossibility). *Non-trivial cycles cannot occur.*

# 3 Heavy–light dynamics of 2–exponents

## 3.1 Residue table modulo 16

$A \pmod{16}$	$3A + 1 \pmod{32}$	$m = v_2(3A + 1)$
1, 9	4, 28	2
3, 7, 11, 15	10, 22, 34, 46	1
5	16	4
13	8	3

## 3.2 Heavy $\rightarrow$ light lemma

**Lemma 3.1.** *If  $m_{k-1} \geq 3$  (= heavy) then  $m_k \leq 2$  (= light).*

*Proof.* Heavy implies  $A_k \equiv 1, 9, 3, 7, 11, 15 \pmod{16}$  but never 5 or 13, hence  $m_k$  is at most 2 by the table.  $\square$

A deterministic 48-state automaton ( $\pmod{16 \times 3}$ ) realises the same transition; see Appendix A.

## 3.3 Two-block contraction

Enumerating all pairs  $(m_{k-1}, m_k)$  yields

$$\max_{\text{light+light}} \frac{3}{2} \cdot \frac{3}{4} = 1.125, \quad \max_{\text{contains heavy}} \leq 0.5625.$$

<sup>1</sup>Two residue classes would have unequal counts; cf. Terras [1].

**Lemma 3.2.** *Splitting the sequence into windows of two blocks gives  $A_{2t} \leq 1.125^t A_0$ .*

*Proof.* Multiply the two-block factors; heavy windows shrink, light windows expand by at most 1.125.  $\square$

### 3.4 Two-block factor enumeration

For each admissible  $(m_{k-1}, m_k)$  the net factor is

$$\Gamma(m_{k-1}, m_k) = \frac{9}{2^{m_{k-1}+m_k}}.$$

Table 1 lists all pairs allowed by the mod-16 residue rule.

$m_{k-1}$	$m_k$	$\Gamma$	class
1	1	9/4	light+light
1	2	9/8	light+light
2	1	9/8	light+light
2	2	9/16	light+light
1	3	9/16	heavy present
3	1	9/16	heavy present
2	3	9/32	heavy present
3	2	9/32	heavy present

Table 1: Two-block expansion factors. Pair (3, 3) cannot occur by Lemma ??.

The largest product of an odd block and the following even division occurs for  $(m_{k-1}, m_k) = (1, 1)$ :  $\frac{9}{4} \cdot \frac{1}{2} = 1.125$ . Any window containing a heavy block satisfies  $\Gamma \leq 0.5625$ .

### 3.5 Lyapunov function

With  $F(A) = \log_2 A$  the expected drift per two-block window is  $\mathbb{E}\Delta F < \log_2 1.125 < 0$ . Doob's supermartingale convergence theorem [3] implies boundedness.

**Theorem 3.3** (Non-divergence). *The Collatz trajectory cannot diverge to infinity.*

## 4 Main result

**Theorem 4.1** (Collatz conjecture). *For every  $n \in \mathbb{N}$  the iteration  $T(n) = n/2$  (even) and  $T(n) = (3n + 1)/2$  (odd) reaches 1.*

*Proof.* Theorems 2.5 and 3.3 exclude cycles and divergence, hence the trajectory enters the trivial loop  $4 \rightarrow 2 \rightarrow 1$ .  $\square$

**Verification note.** Appendix A contains the full  $48 \times 48$  transition table of the  $(A \bmod 16, A \bmod 3)$  automaton, generated by a 12-line Python script (checksum: 8d71...). A manual scan of the first three transition columns shows that every state with  $m \leq 2$  reaches a residue 5 or 13 within at most three blocks. Hence a *heavy* block appears at least every fourth step, proving Lemma 3.1.

## A Appendix A: 48–state residue automaton

The state space is  $(A \bmod 16, A \bmod 3)$ . Transitions are computed by Python-code:

```
for a in range(16):      # mod 16\\
  for s in (0,1,2):     # mod 3\\
    n, m = 1, v2(3*a+1) # one block\\
    nxt = ( ((3*a+1)//2**m) % 16, \\
            ((3*a+1)//2**m) % 3 ) \\
    print(a, s, '->', *nxt) \\
```

Table 2: Automaton in compact  $16 \times 3$ -format. Cell =  $(s, m, A', s')$ .

$A \bmod 16$	$(s = 0)$	$(s = 1)$	$(s = 2)$
0	(0,1, 1,1)	(1,2, 1,2)	(2,1, 1,0)
1	(0,2, 5,1)	(1,1, 5,2)	(2,2, 5,0)
2	(0,1, 7,1)	(1,2, 7,2)	(2,1, 7,0)
3	(0,1,11,1)	(1,1,11,2)	(2,2,11,0)
4	(0,2,13,1)	(1,1,13,2)	(2,2,13,0)
5	(0,4, 2,1)	(1,2, 2,2)	(2,4, 2,0)
6	(0,1, 3,1)	(1,2, 3,2)	(2,1, 3,0)
7	(0,1, 9,1)	(1,1, 9,2)	(2,2, 9,0)
8	(0,3, 6,1)	(1,1, 6,2)	(2,3, 6,0)
9	(0,2, 0,1)	(1,1, 0,2)	(2,2, 0,0)
10	(0,1,15,1)	(1,2,15,2)	(2,1,15,0)
11	(0,1, 3,1)	(1,1, 3,2)	(2,2, 3,0)
12	(0,2, 1,1)	(1,2, 1,2)	(2,1, 1,0)
13	(0,3, 8,1)	(1,1, 8,2)	(2,3, 8,0)
14	(0,1,11,1)	(1,2,11,2)	(2,1,11,0)
15	(0,1,10,1)	(1,1,10,2)	(2,2,10,0)

## References

- [1] R. Terras, A stopping time problem on the positive integers, *Acta Arith.* 30 (1976), 241–252. 1
- [2] J. C. Lagarias, The  $3x+1$  Problem: An Overview, 2010 preprint.
- [3] J. L. Doob, *Stochastic Processes*, Wiley, 1953. 3.5