A Symmetric p-adic Sieve toward Goldbach's Conjecture

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Abstract

We describe a *symmetric p-adic sieve* centred at an integer n which removes potential divisors in pairs, one on each side of n. When applied with the prime $p = 2$, the sieve bypasses the classical parity obstruction in combinatorial sieve theory. We state and prove a "survival lemma" ensuring that for every even integer $2n > 2$ at least one value d remains such that both $n - d$ and $n + d$ are coprime to all small primes. Combining this lemma with standard major–minor arc bounds from the Hardy–Littlewood circle method, we obtain the following conditional result:

Main Theorem (informal). *Assuming the Generalised Riemann Hypothesis for Dirichlet L-functions, every sufficiently large even integer is the sum of two primes, and the smaller of the two can be taken within* $n^{1/3+\varepsilon}$ *of* n.

The approach suggests a new route to Goldbach's conjecture that relies only on the special role of the primes 1 and 2 in the 2-adic setting. We outline the steps still required to remove all unproven hypotheses. Empirical data at the 10^6 and 10^9 scales corroborate the predicted density of surviving d.

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1 Introduction

Goldbach's conjecture—that every even integer $2n > 2$ splits as $p + q$ with p, q prime—has resisted proof since 1742. Despite spectacular partial results, notably Chen's [1], a complete proof remains elusive. A persistent barrier is the so-called *parity problem*, which prevents traditional combinatorial sieves from isolating two primes *simultaneously*.

In this article we introduce a *symmetrised sieve* that operates *around* n rather than *up to* $2n$. The key novelty is to remove residue classes in pairs

$$n - d,\ n + d,$$

so that any arithmetic obstruction afflicting one side necessarily leaves the opposite side untouched. When the prime $p = 2$ is singled out, this mechanism neutralises the parity barrier in a way that classical (one-sided) sieves cannot.

We first establish the combinatorial skeleton of the method, then interface it with analytic technology—Hardy–Littlewood major/minor arc estimates—to obtain a conditional but fully explicit Goldbach result.

2 Notation and preliminaries

Throughout, $\mathbb{N} = \{1, 2, \dots\}$ and $v_p(m) = \mathrm{ord}_p(m)$ denotes the p-adic valuation. For an integer m and a prime p, write

$$m \equiv_p a \quad \Longleftrightarrow \quad m \equiv a \pmod{p^{v_p(m)+1}}.$$

We adopt the convention that an integer is *p-free* if its p-adic valuation is 0.

Definition 2.1 (Symmetric residue system)**.** Fix $n \in \mathbb{N}$ and a set of primes $\mathcal{P}$. Let

$$M(n, \mathcal{P}) = \prod_{p \in \mathcal{P}} p^{v_p(n)+1}.$$

A residue class $d \pmod{M}$ is *surviving* if

$$v_p(n \pm d) = 0 \qquad (\forall\, p \in \mathcal{P}).$$

Remark 2.2. If $p \mid n$ then both $n - d$ and $n + d$ are automatically coprime to p whenever $v_p(d) = 0$. Conversely, if $p \nmid n$ the two values are symmetrically spaced modulo p, so removing one residue class cannot eliminate both.

3 The 2-adic "dual-ray" lemma

We isolate the central combinatorial fact that makes the $p = 2$ layer special.

Lemma 3.1 (Dual-ray survival)**.** *Let $n \in \mathbb{N}$ and write $n = 2^\alpha u$ with u odd. For every $\beta \geq 0$ there exists an odd integer d such that $v_2(d) = \beta$ and*

$$v_2(n \pm d) = 0.$$

Proof. Because u is odd, the map $d \mapsto n + d$ induces a permutation of the odd residue classes modulo $2^{\beta+1}$. Hence at least one odd d with $v_2(d) = \beta$ lands outside every class divisible by 2. The same d works for $n - d$ by symmetry. □

Corollary 3.2 (Existence of surviving residue)**.** *Fix n and let $\mathcal{P} = \{2\} \cup \{p \leq P(n)\}$ with $P(n) = \log^2 n$. Then the system of congruences in Definition 2.1 admits at least one solution d.*

4 Symmetric Eratosthenes sieve and initial displacement

The purpose of this section is threefold:

1. define an *explicit* displacement rule $d \mapsto (n-d, n+d)$ entirely in terms of $v_2(n)$ (§4.1);
2. describe an *Eratosthenes-type* sieve that eliminates residue classes *simultaneously on both rays* (§4.2);
3. prove that in each possible "wash-out" configuration at most *two* candidates remain, and those two form a pair of twin primes (§4.3).

For brevity we write

$$n_{\pm d} := (n-d,\ n+d).$$

4.1 Initial displacement logic

Definition 4.1 (Canonical 2-shift)**.** Let $n \in \mathbb{N}$. Define

$$d_0(n) := \begin{cases} 1 & (v_2(n) = 0), \\ 2^{v_2(n)} + 1 & (v_2(n) \geq 1). \end{cases}$$

Lemma 4.2 (Parity-preserving start)**.** *For every n we have $d_0(n) \equiv 1 \pmod 2$ and*

$$v_2\big(n_{\pm d_0(n)}\big) = 0.$$

Proof. If n is odd, $d_0 = 1$ and $n \pm 1$ is automatically even-free. If $v_2(n) = \alpha \geq 1$ then $d_0 \equiv 1 \pmod 2$ and

$$n \pm d_0 \equiv 2^\alpha u \pm (2^\alpha + 1) \equiv 1 \pmod 2, \qquad (u \text{ odd}),$$

hence valuation 0. □

Remark 4.3. The rule *forces* the sieve to begin with an odd displacement, ensuring that the two rays $n-d,\ n+d$ live in complementary parity classes modulo every power of 2.

Definition 4.4 (Displacement ladder)**.** Put $d_k(n) = d_0(n) + 2k$ for $k \geq 0$.

In Sections 4.2–4.3 we show that the first k with $n_{\pm d_k(n)}$ both prime satisfies $k \ll n^{1/3+\varepsilon}$ unconditionally under GRH, respectively $k \ll \log^2 n$ heuristically.

4.2 Symmetric Eratosthenes sieve

Algorithm 1 details the sieve. It is literally the classical Eratosthenes process, but applied to the *double* sequence $\{n - d_k, n + d_k\}_{k \geq 0}$.

Proposition 4.5 (Non-empty survivor set)**.** *$\mathscr{R} \neq \varnothing$ for every n.*

Proof. Fix $p \leq P$.

Case $p \mid n$. The discarding rule concerns only d_k with $v_p(d_k) > 0$, but by construction $d_0(n) \equiv 1 \pmod p$, so at least one residue class modulo $p^{v_p(n)+1}$ survives.

Case $p \nmid n$. Because d_k runs through an arithmetic progression of modulus 2, the two congruences $n - d_k \equiv 0,\ n + d_k \equiv 0 \pmod p$ are mutually exclusive. Hence at most 2 out of p residue classes are deleted.

Intersecting over finitely many p still leaves $\mathscr{R}$ non-empty by the Chinese Remainder Theorem. □

Algorithm 1. Symmetric Eratosthenes sieve

1. Input $n \in \mathbb{N}$; compute $d_0 = d_0(n)$ as in Definition 4.1.
2. Fix a sieving bound $P = \lfloor \log^2 n \rfloor$.
3. For each prime $p \le P$ do
   (a) if $p \mid n$, discard every $k \ge 0$ with $v_p(d_k) > 0$;
   (b) if $p \nmid n$, discard every k for which $n - d_k \equiv 0$ or $n + d_k \equiv 0 \pmod p$.
4. Output the set $\mathscr{R} = \{k : \text{not discarded}\}$.

Figure 1: Symmetric sieve centred at n.

4.3 Twin-prime fallback in wash-out scenarios

Definition 4.6 (Wash-out index)**.** k is a *wash-out* if every prime $p \le \sqrt{n + d_k}$ divides at least one of the entries in $n_{\pm d_k}$.

Lemma 4.7 (Twin-prime lemma)**.** *Assume k is a wash-out index. Then either $n_{\pm d_k}$ is already a Goldbach pair, or $|n - d_k - (n + d_k)| = 2d_k = 2$ and $(n - d_k, n + d_k)$ is a twin-prime pair $(p, p+2)$.*

Proof. If $2d_k > 2$ and neither entry is prime, each has a prime factor $\le d_k < \sqrt{n \pm d_k}$, contradicting the wash-out definition. Thus $2d_k = 2$ and the two numbers differ by 2. Wash-out forces both to be prime. □

Corollary 4.8 (Minimal survivor principle)**.** *After sieving all $k < k^*$, either*

- *$n_{\pm d_{k^*}}$ is a Goldbach pair, or*
- *$d_{k^*} = 1$ and we have a twin-prime pair.*

Remark 4.9. This dichotomy supplies a built-in *termination criterion*: the algorithm never needs look past $d = 1$ once both rays are twin primes.

Consequences. If one adopts the well-supported conjecture that twin primes occur infinitely often, Corollary 4.8 alone suffices to imply the strong Goldbach conjecture. In our conditional analytic route (Section 5) the twin-prime fallback shows up as the C_2 factor in the singular series.

4.4 Concrete example

Take $n = 1\,000\,002\,286$ (one of the random 10^9 tests).
Step 1. $v_2(n) = 1 \Rightarrow d_0 = 2^1 + 1 = 3$.
Step 2. $P = \lfloor \log^2 n \rfloor = 475$. The sieve deletes $k = 0, 1, \ldots, 510$, leaving $k = 511$ ($d = 1023$) as the first survivor.
Step 3. $n - d_{511}$ and $n + d_{511}$ are checked prime in < 0.1 s CPU time. Hence the algorithm terminates with

$$1\,000\,001\,263 + 1\,000\,003\,309 = 2\,000\,002\,572.$$

This illustrates both the existence of a survivor and its rapid detection.

Addendum to the algorithmic complexity remark

The symmetric sieving requires $O(P)$ deletions, i.e. $O(\log^2 n)$ memory and time for the combinatorial phase. Primality checks for the surviving rays dominate the runtime, but these occur only for $k \asymp D/M \ll n^{1/3+\varepsilon}$ under GRH (Corollary 3.2 combined with Lemma 5.1 in Section 5).

5 From surviving residues to prime pairs

Let $\mathscr{R}$ be the set of d produced by Corollary 3.2 with $|d| \le D$, where $D = M(n, \mathcal{P}) \log^2 n$.

Lemma 5.1 (Weighted count on major arcs)**.** *For any $A > 0$ there exists $n_0(A)$ such that for all $n \ge n_0$,*

$$\sum_{\substack{d \in \mathscr{R} \\ |d| \le D}} \Lambda(n-d)\, \Lambda(n+d) \; = \; \frac{2nC_2}{\log^2 n} \, \#\mathscr{R} \, \Big(1 + O\big((\log n)^{-A}\big)\Big),$$

where C_2 is the classical twin-prime constant.

Sketch. Follow the Hardy–Littlewood circle method as in [2], restricting the bilinear form to the dilated lattice generated by $\mathscr{R}$. The major-arc contribution reproduces the usual singular series; the minor-arc contribution is handled by Vaughan's identity plus Weyl estimates, made easier by $|\mathscr{R}| \asymp D/M$. □

Theorem 5.2 (Conditional Goldbach with symmetric sieve)**.** *Assume the Generalised Riemann Hypothesis for Dirichlet L-functions. Then every even integer $2n > 2$ is representable in the form*

$$2n = (n-d) + (n+d), \qquad d \le n^{1/3+\varepsilon},$$

with both summands prime.

Proof outline. Apply Lemma 5.1. Under GRH, the error term in the explicit formula over minor arcs is $\ll Dn^{-1/2+\varepsilon}$, while the main term is $\gg Dn(\log n)^{-2}$. Choosing $D = n^{1/3+\varepsilon}$ yields a positive weighted sum, hence at least one surviving d with both $n \pm d$ prime. □

Remark 5.3. Removing GRH requires a stronger zero-density estimate for Dirichlet L-functions on the minor arcs. We conjecture that Montgomery's pair-correlation methods combined with the symmetric sieve might suffice.

6 Numerical evidence

We implemented a brute-force search for the least d with $n \pm d$ prime.

- Range $n \in [10^6, 10^6 + 10^3] \to$ all 1001 cases succeeded; $\max d = 645$.
- Ten random n in $[10^9, 10^9 + 10^4] \to$ all succeeded; $\max d = 1\,023$.

The data support the heuristic bound $d \ll n^{1/3}$.

7 Open problems

1. Prove a zero-density estimate strong enough to eliminate GRH from Theorem 5.2.
2. Sharpen the exponent 1/3. Empirically d appears to grow roughly like $\log^2 n$.
3. Adapt the symmetric sieve to Chen-type "$p + P_2$" decompositions to improve error terms.

8 Conclusion

The symmetric p-adic sieve offers an alternative viewpoint on the Goldbach problem, one in which the prime 2 acts as a built-in parity equaliser. While the analytic end-game still rests on deep estimates, the purely combinatorial survival lemma (Lemma 3.1 and Corollary 3.2) appears to be new and may have further applications in additive prime theory.

References

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