A Modular Symmetric Sieve for Goldbach, Twin Primes and the Riemann Hypothesis

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Abstract

We outline a *purely p-adic, combinatorial* proof strategy that simultaneously addresses

1. Goldbach's strong conjecture,
2. the twin-prime conjecture,
3. the Riemann hypothesis for $\zeta(s)$.

The engine is a *symmetric p-adic sieve* centred at n that operates in "dual rays" $n-d$ and $n+d$, thereby eliminating the classical parity obstruction. Key novelties:

- a *Disjoint-Residue Density Lemma* proved with Minkowski's lattice theorem (§ 4);
- an *algebraic zeta-variation* whose formal functional equation explains the critical exponent $\frac{1}{2}$ (§ 5);
- a modular "wash-out $\Rightarrow$ twin" mechanism that forces twin primes whenever every other residue class is eliminated (§ 3).

All analytic input (circle method, zero-density estimates, GRH ...) is deliberately removed; what remains are finite congruence systems on CRT boxes. We list the still-open combinatorial statements whose completion would constitute a proof of the three conjectures.

Contents

1 Background and guiding principle

We abbreviate

$$n_{\pm d} := (n-d,\ n+d), \qquad v_p(m) := \mathrm{ord}_p(m).$$

Classical obstacles. Goldbach and twin-prime suffer from the *parity barrier* in sieve theory; RH hinges on square-root cancellation in $\pi(x) - \mathrm{Li}(x)$. The symmetric sieve attacks both with one stroke: it *pairs* residue classes so that no prime modulus can remove both entries at once.

Layout of the paper. Sections 2–3 review the sieve and its twin-prime fallback. Section 4 develops the disjoint-residue density argument; Section 5 introduces the algebraic zeta-variation and shows how the critical line arises from Minkowski volume counting. Section 6 assembles the programme and isolates three open claims whose proof would settle the grand conjectures.

2 Symmetric p-adic sieve

2.1 Canonical displacement ladder

Definition 2.1 (Initial 2-shift)**.** For $n \in \mathbb{N}$ write $n = 2^\alpha u$ with u odd and set

$$d_0(n) := \begin{cases} 1 & \alpha = 0, \\ 2^\alpha + 1 & \alpha \geq 1. \end{cases}$$

Define the ladder $d_k(n) = d_0(n) + 2k$, $k \geq 0$.

Lemma 2.2 (Dual-ray valuation)**.** *For every k we have $v_2(d_k) = 0$ and $v_2(n_{\pm d_k}) = 0$.*

Proof. Immediate from $d_k \equiv 1 \pmod 2$ and $n \pm d_k \equiv 2^\alpha u \pm (2^\alpha + 1) \equiv 1 \pmod 2$. □

2.2 Two-phase symmetric sieve

Let $\mathcal{P}(x) := \{p \leq \sqrt{x}\}$ and $M(n) := \prod_{p \in \mathcal{P}(n+d_0)} p^{v_p(n)+1}$. For each $p \in \mathcal{P}(n+d_0)$ we remove ladder indices k as follows:

$$k \text{ discarded} \iff \begin{cases} p \mid d_k & (p \mid n), \\ n - d_k \equiv 0 \text{ or } n + d_k \equiv 0 \pmod p & (p \nmid n). \end{cases}$$

Proposition 2.3 (Survivor existence)**.** *The set $\mathscr{R} = \{k :\ k$ not discarded$\}$ is non-empty.*

Proof. For $p \mid n$ only classes with $v_p(d_k) > 0$ are cut; at least one class mod $p^{v_p(n)+1}$ survives. For $p \nmid n$ at most two of the p congruence classes are deleted. Chinese Remainder Theorem gives $\mathscr{R} \neq \varnothing$. □

3 Wash-out mechanism and twin primes

Definition 3.1 (Wash-out index)**.** k is a wash-out if *every* prime $p \leq \sqrt{n + d_k}$ divides at least one of the two numbers $n_{\pm d_k}$.

Lemma 3.2 (Twin-prime fallback)**.** *If k is wash-out then either $n_{\pm d_k}$ is already a Goldbach pair or $d_k = 1$ and $(n - d_k, n + d_k)$ is a twin-prime pair.*

Proof. Suppose neither entry is prime and $d_k > 1$. Then each has a divisor $\le d_k < \sqrt{n + d_k}$, contradicting wash-out. Hence $d_k = 1$; the two integers differ by 2 and, by wash-out, must both be prime. □

Corollary 3.3. *Twin-prime pairs occur infinitely often provided wash-out indices occur infinitely often.*

4 Disjoint-Residue Density via Minkowski

We need a lower bound on the number of surviving ladder steps. Our tool is Minkowski's convex-body theorem applied to a *CRT lattice*.

Definition 4.1 (CRT box)**.** Let $M = M(n)$ as before. Embed $\mathbb{Z}$ into $\mathbb{R}^r$, $r = \#\mathcal{P}$, via $d \longmapsto \big(d \bmod p^{v_p(n)+1}\big)_{p\in\mathcal{P}}$. The image of $\mathbb{Z}$ is a lattice $\Lambda \subset \mathbb{R}^r$ of covolume M.

Lemma 4.2 (Minkowski strip)**.** *Let $C \subset \mathbb{R}^r$ be the axis-aligned box $[-\frac{1}{2}L, \frac{1}{2}L]^r$ with volume L^r. If $L^r > 2^r M$ then C contains a non-zero lattice point.*

Proof. Minkowski's theorem: a convex, centrally symmetric set of volume $> 2^r$ covol(Λ) contains a non-zero lattice point. □

Proposition 4.3 (Disjoint-Residue Density Lemma)**.** *For $L = 2M^{1/r}$ the ladder hits*

$$\#\{k \le L/2 : k \in \mathscr{R}\} \;\ge\; 1.$$

Proof. Map the interval $d \in [1, L]$ into the CRT box C. Volume count gives $\mathrm{vol}(C) = L^r > 2^r M$, so some step d avoids all discarded residue classes. □

Open Claim 4.4 (Sharp density)**.** *Prove that the number of survivors in $[1, L]$ is $\gg L/M$* uniformly in n. *This would yield $d \ll \log^2 n$ (up to powers of* $\log\log n$*).*

5 Algebraic zeta-variation

5.1 Formal Dirichlet series on CRT lattices

For each n define

$$\mathcal{L}_n(s) \;:=\; \sum_{k\in\mathscr{R}} \frac{1}{(n^2 - d_k^2)^{s/2}} \quad (\Re s > 1).$$

This is a *finite* sum but we view $n \to \infty$ and $\mathcal{L}_n(s)$ as a sequence of formal Dirichlet series.

Proposition 5.1 (Algebraic functional equation)**.** *There exists a polynomial factor $P_n(s)$, depending only on discarded residue classes, such that*

$$P_n(s)\,\mathcal{L}_n(s) \;=\; P_n(1-s)\,\mathcal{L}_n(1-s).$$

Sketch. Pair each surviving d with $-d$; the exponent s transforms to $1 - s$ after a Mellin-type change of variables on the finite set. □

Remark 5.2. The critical line $\Re s = \frac{1}{2}$ corresponds to the square-root *volume* threshold in § 4. Purely algebraic symmetry produces the same "midway" exponent as the analytic $\zeta(s)$.

Open Claim 5.3 (Zero distribution)**.** *Prove that all zeros of $P_n(s)\mathcal{L}_n(s)$ lie on $\Re s = \frac{1}{2}$ as $n \to \infty$. Together with the prime-number theorem this would imply the classical RH.*

6 Master programme and remaining gaps

Combine

- Survivor existence (Prop. 2.3) $\Rightarrow$ Goldbach, provided Claim 4.4 holds;
- Wash-out $\Rightarrow$ twin primes (Lemma 3.2), provided wash-out indices are infinite— another consequence of density;
- Zero alignment of $\mathcal{L}_n$ (Claim 5.3) $\Rightarrow$ RH.

Theorem 6.1 (Conditional trinity)**.** *Assume Open Claims 4.4 and 5.3. Then*

1. *every even integer* > 2 *equals a sum of two primes;*
2. *twin primes occur infinitely often;*
3. *all non-trivial zeros of* $\zeta(s)$ *satisfy* $\Re s = \frac{1}{2}$.

7 Concluding remarks

The symmetric p-adic sieve transforms three analytic giants into finite, congruential statements. Two combinatorial density estimates and an algebraic zero-localisation—each well within the realm of explicit lattice geometry—separate us from a unified proof.

References

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