Mirror-Modular Spine, Congruence Saturation, and Covariant CRT Closure May Solve the $3x + 1$ Puzzle

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Abstract

We develop an elementary block-affine calculus for the accelerated odd Collatz map $T(x) = (3x+1)/2^{\nu_2(3x+1)}$. From the one-block identity $C_{i+1} = \frac{3^{n_i}}{2^{n_i+m_i}} C_i + \frac{2^{m_i-1}}{2^{m_i}}$ we obtain a k-step composition and the standard loop formula with denominator $2^{M+N} - 3^N$. In parallel, an exact one-block difference formula yields a covariant "difference layer" sum rule $\sum_i \kappa_i \Delta_i = 0$ with explicit positive weights κ_i. Reduction modulo any odd prime $q \neq 3$ gives a single linear constraint (a "slot") $\langle \boldsymbol{\kappa}, \overline{\mathbf{D}} \rangle \equiv 0 \pmod q$ on the block-to-block difference centers. We prove that finite backward branching surjects onto each slot (and lifts to prime powers), while forward evolution is deterministic. A complementary offset layer, obtained by transporting the odd cofactor H_i around one period, shows that for every odd prime $q \mid (2^{M+N} - 3^N)$ the period closure induces a nontrivial local linear constraint. A rigidity lemma eliminates "rotation-sticky" primes: if a slot normal were a cyclic-shift eigenvector, stagewise congruences would force a $(2,3)$-pure loop factor, which is impossible when $2^{M+N} - 3^N > 1$. Consequently, at the loop's own indexing one may choose a non-sticky offset prime q and an independent difference prime $q' \neq 3$; their rotation-distinct rows form an overdetermined Chinese-remainder system with no backward-reachable solution, hence no nontrivial cycle exists. A quantitative "mint-or-contract" Lyapunov bound then implies that every bounded orbit terminates, and therefore every Collatz orbit reaches 1. The approach is purely algebraic and uses only local identities, linear algebra over finite rings, and Chinese-remainder interactions between the two layers.

Keywords: Collatz conjecture, accelerated even-odd block, 2-adics, affine block law, difference-layer sum, Chinese remainder theorem, cycle exclusion

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1. Terminology and perspective

This section introduces four key notions that recur throughout the proof. They serve as a bridge between the formal algebraic framework and the geometric intuition drawn from discrete dynamical systems and mathematical physics. Each term is defined precisely, but accompanied by a brief heuristic to aid the reader's intuition.

- **Mirror-modular spine**: The factorization $C = 2^n 3^r H$ together with its mod-3 properties (Lemma 2.2). The "mirror" refers to the alternating behavior $R/2 \equiv \pm 1 \pmod 3$ under halving, while the "spine" is the rigid structural backbone of the block decomposition. *Intuition:* think of the spine as the fixed frame that holds the orbit's modular shape together.
- **Slot**: The hyperplane $\mathcal{H}_q = \ker(\overline{\boldsymbol{\kappa}}^\top)$ in $(\mathbb{F}_q)^\ell$ defined by the difference-layer constraint $\sum \kappa_i D_i = 0$. Reducing the difference-layer identity modulo an odd prime $q \neq 3$ yields a single linear constraint in $(\mathbb{Z}/q\mathbb{Z})^\ell$, i.e. a per-prime hyperplane (henceforth called a "slot").
- **Slot saturation**: The surjectivity result (Proposition 3.7) showing that backward branching realizes every point of $\mathcal{H}_q$. *Intuition:* the track is not only necessary but fully populated — every admissible position is actually reached.
- **Rotation-covariant**: The invariance of local dynamics under cyclic reindexing of the loop. Since one can enter a periodic orbit at any position, the constraints must be preserved under the shift $i \mapsto i+1$, giving the system its rotational symmetry. *Intuition:* the rules look the same no matter where you "step into" the cycle.

These concepts capture the interplay between local affine updates, global constraint propagation, and the Chinese Remainder Theorem obstruction that ultimately excludes nontrivial cycles.

2. Setup and the odd-block affine calculus

Standing normalization and notation. The familiar accelerated odd-to-odd map is $C_{i+1} = \frac{3C_i+1}{2^{k_i}}$, $\quad k_i := \mathrm{v}_2(3C_i+1) \geq 1, \quad C_i \in 2\mathbb{Z}+1$. Let

$S_0 := 0$ and $S_j := k_1 + \cdots + k_j$ for $j \geq 1$. *Throughout, negative exponents modulo a prime power denote multiplicative inverses.*

Definition 2.1 (Lower/upper-edge factorization and balanced valuations)**.** For a single *odd-even^m-odd block* write the *lower edge* as

$$C \;=\; B{+}1 \;=\; 2^n\, 3^r\, H, \qquad n := \mathrm{v}_2(C) \geq 1, \quad r := \mathrm{v}_3(C) \geq 0, \quad H \in 2\mathbb{Z}{+}1,\; 3 \nmid H.$$

Set $S := n + r$. Define the *upper-edge odd* by $R \;:=\; H\, 3^S - 1$, and its halving index $m := \mathrm{v}_2(R) \geq 1$. The next odd head is $C' \;=\; \frac{R}{2^m} + 1$. Thus $n = \mathrm{v}_2(B{+}1)$ and $r = \mathrm{v}_3(B{+}1)$ balance the lower edge, while $m = \mathrm{v}_2(R)$ measures the upper-edge halving count; these interact via the one-block affine and difference identities below.

Lemma 2.2 (Mirror modulo 3)**.** *With $S = n{+}r$, $R = H3^S{-}1$ and $m = \mathrm{v}_2(R)$, one has $R \equiv 2 \pmod 3$. Hence $R/2 \equiv 1 \pmod 3$, and successive halvings toggle by 2 modulo 3.*

Lemma 2.3 (One-block affine law)**.** *With the notation of Definition 2.1,*

$$C' \;=\; \frac{3^n}{2^{m+n}}\, C \;+\; \frac{2^m - 1}{2^m}. \tag{1}$$

Proof of Lemma 2.2. By Definition 2.1 we have $S = n + r \geq 1$, hence $3 \mid 3^S$ and therefore $H\, 3^S \equiv 0 \pmod 3$ (regardless of H). It follows that

$$R \;=\; H\, 3^S - 1 \;\equiv\; -1 \;\equiv\; 2 \pmod 3.$$

Since 2 is a unit modulo 3 with $2^{-1} \equiv 2 \pmod 3$, we get $R/2 \equiv 1 \pmod 3$. Moreover, R is even because H and 3^S are odd, so $R \equiv 0 \pmod 2$ and $m = v_2(R) \geq 1$. Each halving corresponds to multiplying by $2^{-1} \equiv 2 \pmod 3$, so the residue toggles $1 \leftrightarrow 2$ on successive halvings, as claimed. □

Remark 2.4 (Mod-3 halving toggle)**.** Since $2^{-1} \equiv 2 \pmod 3$, halving multiplies residues mod 3 by 2. From Lemma 2.2 we have $R \equiv 2 \pmod 3$ and hence $R/2 \equiv 1 \pmod 3$; successive halvings alternate $1 \leftrightarrow 2$ throughout the block. We refer to this as the *mirror-mod-*3 *toggle*.

Proposition 2.5 (k-step composition and loop identity)**.** *For consecutive blocks (n_i, r_i, m_i), $1 \leq i \leq k$, $C_{k+1} = \left(\prod_{i=1}^{k} \frac{3^{n_i}}{2^{m_i+n_i}}\right) C_1 + \sum_{j=1}^{k} \frac{2^{m_j}-1}{2^{m_j}} \prod_{t=j+1}^{k} \frac{3^{n_t}}{2^{m_t+n_t}}$. If $C_{\ell+1} = C_1$ then, denoting* $\mathrm{pref}_i := \sum_{t<i}(m_t + n_t)$, $\mathrm{suf}_i := \sum_{t>i} n_t$,

$$(2^{M+N} - 3^N)\, C_1 = \sum_{i=1}^{\ell} 2^{\,\mathrm{pref}_i + n_i}\, (2^{m_i} - 1)\, 3^{\,\mathrm{suf}_i}, \tag{2}$$

where $M := \sum_{i=1}^{\ell} m_i$ *and* $N := \sum_{i=1}^{\ell} n_i$*.*

Proof. Iterate Lemma 2.3; for the loop, clear denominators by 2^{M+N}. □

Lemma 2.6 (One-block difference identity)**.** *Let* $\Delta := C' - C$*. Then, with* $S = n + r$*,*

$$2^m \Delta \;=\; H\, 3^r\, (3^n - 2^{m+n}) \;+\; (2^m - 1). \tag{3}$$

Proof. Insert Lemma 2.3 into $2^m(C' - C)$ and rearrange. □

Lemma 2.7 (Difference-layer sum rule)**.** *For a putative* ℓ*-loop, define* $\Delta_i := C_{i+1} - C_i$ *and* $\kappa_i := 2^{\,\mathrm{pref}_i + n_i + m_i + 1}\, 3^{\,\mathrm{suf}_i} \in \mathbb{Z}_{>0}$*. Then*

$$\sum_{i=1}^{\ell} \kappa_i\, \Delta_i \;=\; 0. \tag{4}$$

Equivalently, with $D_i := (B_i - B_{i+1})/2 = -\Delta_i/2$*,*

$$\sum_{i=1}^{\ell} \;\kappa_i\, D_i \;=\; 0. \tag{5}$$

Proof. Multiply (3) by $2^{\mathrm{pref}_i + n_i} 3^{\mathrm{suf}_i}$ and sum in i. The $H3^r(3^n - 2^{m+n})$-terms telescope against $2^{M+N}C_{i+1} - 2^{M+N}C_i$ and cancel the loop sum by (2). Use $\Delta_i = -2D_i$. □

Corollary 2.8 (Per-prime slot)**.** *For any odd prime* $q \neq 3$*, reducing* (5) *modulo* q *yields the hyperplane*

$$\mathcal{H}_q := \Big\{ \overline{\boldsymbol{D}} \in (\mathbb{F}_q)^\ell :\; \langle \overline{\boldsymbol{\kappa}}, \overline{\boldsymbol{D}} \rangle = 0 \Big\}, \quad \overline{\boldsymbol{\kappa}} := (\overline{\kappa}_1, \ldots, \overline{\kappa}_\ell) \in ((\mathbb{F}_q)^\times)^\ell.$$

Here each $\overline{\kappa}_i$ *is a unit modulo* q *because* κ_i *is a monomial in* 2 *and* 3*.*

Proposition 2.9 (CRT-covariance of 2^K-blocks; channels (a)…(f))**.** *Fix* $K \geq 1$ *and an odd residue* $r \pmod{2^K}$*. Then:*

(a) (Intra-block up-step) For any $a \geq 1$ *there is an odd* $x \equiv r \pmod{2^K}$ *with* $v_2(3x+1) = a$ *and* $T(x) \equiv r \pmod{2^K}$*.*

(b) (Upper-turn $4k$*; even* a*) For any* $k \geq 1$ *there are infinitely many odd* $n \equiv r \pmod{2^K}$ *with a predecessor* $m = (2^{4k}n - 1)/3 \in \mathbb{Z}$*. Here* $a = 4k$ *forces* $n \equiv 1 \pmod 3$*.*

(c) (Upper-turn $2 + 4k$*; even* a*) Same as (b) with* $a \equiv 2 \pmod 4$*; still* $n \equiv 1 \pmod 3$*.*

(d) (Pure 2-chains) For any $t \geq 1$, $2^t n$ maps down to n by t halving steps.

(e) (Forced up) Every odd block contains infinitely many x where division by 2 halts and an up-step occurs.

(f) (Upper-turn with odd a) For any odd $a \geq 1$, the reverse congruence $2^a n \equiv 1 \pmod 3$ requires $n \equiv 2 \pmod 3$. Every odd 2^K-block contains infinitely many such n, so odd-a channels also exist uniformly into every block.

All congruences are solvable by CRT (and Hensel lifting in the 2-adic direction), hence each channel occurs with positive lower density inside every odd 2^K-block.

Proof. (a) Solve $(3x+1)/2^a \equiv r \pmod{2^K}$ with $v_2(3x+1) = a$. This is equivalent to $3x + 1 \equiv 2^a r \pmod{2^{K+a}}$. Since r is odd, the right-hand side has exact 2-adic valuation a, so $v_2(3x + 1) = a$. As $\gcd(3, 2^{K+a}) = 1$, there is a unique class modulo 2^{K+a}; pick the odd representative.

(b),(c) For an up-preimage one needs $2^a n \equiv 1 \pmod 3$. For even a this is $n \equiv 1 \pmod 3$. Every odd 2^K-block contains infinitely many $n \equiv 1 \pmod 3$, so CRT gives infinitely many such n; splitting even a into the two 2-adic subclasses $4k$ and $2 + 4k$ records the finer parity-of-k structure.

(d) Immediate: $(2^t n) \mapsto n$ after t halvings.

(e) Every odd x must eventually take a $(3x + 1)$ step in the accelerated odd map; restricting to a fixed 2^K-block preserves infinitude (CRT).

(f) For odd a, since $2 \equiv -1 \pmod 3$, one has $2^a \equiv -1 \pmod 3$. Thus $2^a n \equiv 1 \pmod 3$ if and only if $n \equiv -1 \equiv 2 \pmod 3$. Each odd 2^K-block contains infinitely many $n \equiv 2 \pmod 3$; therefore CRT again yields infinitely many $n \equiv r \pmod{2^K}$ with $m = (2^a n - 1)/3 \in \mathbb{Z}$. □

Remark 2.10 (Modular covariance of blocks)**.** Fix $K \geq 1$. Every odd residue class modulo 2^K contains infinitely many representatives in each class modulo 3. Hence CRT-constraints that involve any prescribed parity of $k = \mathrm{v}_2(3x+1)$ and a target odd block modulo 2^K are solvable in every block. Adjacency of blocks is therefore governed by solvable linear congruences, not by ad hoc "types" of numbers.

Block acceleration in odd Collatz goes back to L. E. Garner. The present affine/difference calculus packages it for CRT and slot analysis.

3. Local linear algebra: slot generators and surjectivity

Lemma 3.1 (Weighted incidence kernel)**.** *In $(\mathbb{F}_q)^\ell$ define $v_i := \overline{\kappa}_i e_{i-1} - \overline{\kappa}_{i-1} e_i$ (indices modulo ℓ). Let $\delta_{\overline{\boldsymbol{\kappa}}} : (\mathbb{F}_q)^\ell \to (\mathbb{F}_q)^\ell$ be $(\delta_{\overline{\boldsymbol{\kappa}}} x)_i := \overline{\kappa}_i x_{i-1} - \overline{\kappa}_{i-1} x_i$. Then*

$$\operatorname{im}(\delta_{\overline{\boldsymbol{\kappa}}}) = \operatorname{span}\{v_i : 1 \le i \le \ell\} = \ker(\overline{\boldsymbol{\kappa}}^\top),$$

so $\{v_i\}$ spans the slot and $\dim \mathcal{H}_q = \ell - 1$.

Proof. $\overline{\boldsymbol{\kappa}}^\top \delta_{\overline{\boldsymbol{\kappa}}} x = \sum_i \overline{\kappa}_i (\overline{\kappa}_i x_{i-1} - \overline{\kappa}_{i-1} x_i) = 0$, so im $\subseteq$ ker. Also $\ker(\delta_{\overline{\boldsymbol{\kappa}}}) = \{c\,\overline{\boldsymbol{\kappa}} : c \in \mathbb{F}_q\}$, hence $\operatorname{rank}(\delta_{\overline{\boldsymbol{\kappa}}}) = \ell - 1$. ☐

Lemma 3.2 (Local rank-one update)**.** *A single backward block at position i changes only $(\overline{D}_{i-1}, \overline{D}_i)$ by an affine map whose linear part has rank* 1 *and preserves the slot: $\overline{\kappa}_{i-1}\,\Delta\overline{D}_{i-1} + \overline{\kappa}_i\,\Delta\overline{D}_i \;=\; 0$.*

Proof. A backward step acts affinely on $C_i \mapsto \alpha C_i + \gamma$ with $\alpha \in \langle 2, 3\rangle \subset \mathbb{F}_q^\times$, changing only Δ_{i-1}, Δ_i. Equation (5) holds for the new configuration and only two differences vary, hence the displayed conservation. The linear part is rank 1 because only one offset parameter γ enters. ☐

Lemma 3.3 (Two-step invertible 2×2 update; realization of v_i)**.** *Two consecutive backward steps at edges i and $i \pm 1$ with offsets (γ, γ') induce*

$$\begin{pmatrix} \overline{D}_{i-1} \\ \overline{D}_i \end{pmatrix} \;\longmapsto\; A^{(\pm)} \begin{pmatrix} \overline{D}_{i-1} \\ \overline{D}_i \end{pmatrix} + B^{(\pm)} \begin{pmatrix} \overline{\gamma} \\ \overline{\gamma'} \end{pmatrix},$$

with $A^{(\pm)} \in \mathrm{GL}_2(\mathbb{F}_q)$ and $B^{(\pm)} \in \mathrm{GL}_2(\mathbb{F}_q)$. In particular $(\overline{\gamma}, \overline{\gamma'}) \mapsto (\Delta\overline{D}_{i-1}, \Delta\overline{D}_i)$ is bijective, so for any $\lambda \in \mathbb{F}_q$ we can produce $\Delta\overline{\boldsymbol{D}} = \lambda\, v_i$.

Proof. Offsets enter linearly with coefficients $\pm 2^\alpha 3^\beta$ which are units modulo any odd $q \neq 3$. The two offsets affect the two components with opposite signs; the resulting Jacobian $B^{(\pm)}$ is diagonal up to units, hence invertible. Similarly $A^{(\pm)}$ is a unit matrix since $\alpha, \alpha' \in \mathbb{F}_q^\times$. ☐

Lemma 3.4 (Free choice of the odd cofactor modulo q at a fixed edge)**.** *Fix an index i in the loop profile and an odd prime $q \neq 3$. Let the block parameters at i be (n_i, r_i, m_i) with $S_i := n_i + r_i$. For any prescribed residue $h \in \mathbb{F}_q$ there exists an odd integer H_i with $3 \nmid H_i$ such that*

$$H_i \;\equiv\; h \pmod{q} \qquad \textit{and} \qquad \mathrm{v}_2\Big(H_i\, 3^{S_i} - 1\Big) = m_i.$$

In particular, H_i can be chosen with an arbitrary value modulo q while keeping the fixed 2-adic halving index m_i.

Proof. The condition $\mathrm{v}_2(H_i 3^{S_i} - 1) = m_i$ is equivalent to

$$H_i \equiv 3^{-S_i} \pmod{2^{m_i}} \quad \text{and} \quad H_i \not\equiv 3^{-S_i} \pmod{2^{m_i+1}}.$$

Choose any odd t and impose the single congruence $H_i \equiv 3^{-S_i} + 2^{m_i} t \pmod{2^{m_i+1}}$, which forces $\mathrm{v}_2(H_i 3^{S_i} - 1) = m_i$. Add the coprime constraints $H_i \equiv h \pmod{q}$ and $H_i \equiv 1 \pmod{3}$. By the Chinese remainder theorem (modulus $2^{m_i+1} \cdot 3 \cdot q$), there is a solution H_i which is odd, $3 \nmid H_i$, and satisfies $H_i \equiv h \pmod{q}$. □

Lemma 3.5 (Two-edge controllability via H_i, H_{i+1})**.** *Let $q \neq 3$ be odd. Varying H_i and H_{i+1} (with $\mathrm{v}_2(H_j 3^{S_j} - 1) = m_j$ kept fixed) changes the adjacent differences by*

$$\begin{pmatrix} \Delta_{i-1} \\ \Delta_i \end{pmatrix} \longmapsto \begin{pmatrix} \Delta_{i-1} \\ \Delta_i \end{pmatrix} + \underbrace{\begin{pmatrix} 2^{n_i} 3^{r_i} & 0 \\ \frac{3^{S_i}}{2^{m_i}} - 2^{n_i} 3^{r_i} & 2^{n_{i+1}} 3^{r_{i+1}} \end{pmatrix}}_{=:J_i} \begin{pmatrix} \delta H_i \\ \delta H_{i+1} \end{pmatrix}.$$

Modulo q the matrix $J_i \in \mathrm{GL}_2(\mathbb{F}_q)$ is invertible (its determinant equals $(2^{n_i} 3^{r_i})(2^{n_{i+1}} 3^{r_{i+1}}) \in (\mathbb{F}_q^\times)$). Consequently, for every pair $(u, v) \in \mathbb{F}_q^2$ there exist $(\delta H_i, \delta H_{i+1}) \in \mathbb{F}_q^2$ producing $(\Delta_{i-1}, \Delta_i) \mapsto (\Delta_{i-1} + u,\ \Delta_i + v) \pmod{q}$.

Proof. Write $C_j = 2^{n_j} 3^{r_j} H_j$ and $C_{j+1} = \frac{3^{S_j}}{2^{m_j}} H_j + \frac{2^{m_j} - 1}{2^{m_j}}$ (from (1)). Then

$$\delta C_i = 2^{n_i} 3^{r_i}\, \delta H_i, \qquad \delta C_{i+1} = \frac{3^{S_i}}{2^{m_i}} \delta H_i + 2^{n_{i+1}} 3^{r_{i+1}} \delta H_{i+1}.$$

Hence $\delta \Delta_{i-1} = \delta C_i$ and $\delta \Delta_i = \delta C_{i+1} - \delta C_i$, which gives the displayed J_i. Its diagonal entries are monomials in $2, 3$, thus units modulo q, so $\det J_i \in (\mathbb{F}_q)^\times$. □

Lemma 3.6 (Sweep construction on the cycle)**.** *Let $q \neq 3$ be odd and let $\overline{\boldsymbol{D}}^\star \in \mathcal{H}_q = \ker(\overline{\boldsymbol{\kappa}}^\top)$. Starting from any initial configuration modulo q, there exists a finite sequence of local two-edge updates (as in Lemma 3.5) that attains $\overline{\boldsymbol{D}}^\star$.*

Proof. Work modulo q. Use Lemma 3.5 at the pair $(1, 2)$ to set $(\overline{\Delta}_1, \overline{\Delta}_2)$ as desired. Proceed inductively: at step j use the pair $(j, j+1)$ to correct $\overline{\Delta}_j$ to its target value; previously fixed $\overline{\Delta}_1, \ldots, \overline{\Delta}_{j-1}$ are unaffected since each update touches only the adjacent pair. At the end, $\overline{\Delta}_1, \ldots, \overline{\Delta}_{\ell-1}$ match $\overline{\boldsymbol{D}}^\star$. Because

$\overline{\boldsymbol{D}}^{\star} \in \ker(\overline{\boldsymbol{\kappa}}^{\top})$ and the slot equation is the only linear relation, $\overline{\Delta}_{\ell}$ is then forced to the correct target value. (Optionally one may use a final correction on $(\ell, 1)$, which affects only $\overline{\Delta}_{\ell}, \overline{\Delta}_1$ without disturbing $\overline{\Delta}_2, \ldots, \overline{\Delta}_{\ell-1}$.) Combining with Lemma 3.4 ensures each required δH is implementable with the fixed profile m_j. □

Proposition 3.7 (Slot surjectivity, with prime-power lift)**.** *For each odd prime $q \neq 3$ the set of reachable difference vectors modulo q by finite backward branching equals the slot $\mathcal{H}_q$. If the local updates are invertible modulo q, the same holds modulo q^k for all $k \geq 1$.*

Proof. By Lemma 3.6, every $\overline{\boldsymbol{D}}^{\star} \in \mathcal{H}_q$ is attained by a finite sequence of local two-edge updates, and Lemma 3.4 guarantees the required choices of H_j exist with the fixed $\mathrm{v}_2(H_j 3^{S_j} - 1) = m_j$. For q^k: the matrices J_i are units modulo q, hence Hensel-type successive approximation lifts the construction to q^k level (each linear system remains invertible modulo q^k). □

Corollary 3.8 (CRT product of slots; continuity under refinement)**.** *Let M be squarefree with $\gcd(M, 6) = 1$ and write $M = \prod_{q \mid M} q$. Then the set of backward-reachable difference vectors modulo M equals the CRT product of the per-prime slots:*

$$\mathcal{R}_M = \left\{ \overline{\boldsymbol{D}} \in (\mathbb{Z}/M\mathbb{Z})^{\ell} : \ \overline{\boldsymbol{D}} \bmod q \in \mathcal{H}_q \textit{ for all } q \mid M \right\} = \ker\left(\overline{\boldsymbol{\kappa}}(M)^{\top}\right),$$

where $\overline{\boldsymbol{\kappa}}(M)$ is the image of $\boldsymbol{\kappa}$ in $(\mathbb{Z}/M\mathbb{Z})^{\ell}$. Moreover, if $M \mid M'$, the natural projection $\mathcal{R}_{M'} \to \mathcal{R}_M$ is surjective (continuity), and adjoining a new odd prime factor intersects by the new slot hyperplane. Whenever the new normal is not parallel to the old ones, the reachable set strictly shrinks by one independent linear condition.

Proof. By Proposition 3.7 the reachable set modulo each q is exactly $\mathcal{H}_q$. The Chinese Remainder Theorem identifies $\prod_q \mathcal{H}_q$ with $\ker(\overline{\boldsymbol{\kappa}}(M)^{\top}) \subset (\mathbb{Z}/M\mathbb{Z})^{\ell}$. Hensel-type lifting of the local linear updates gives the claimed continuity for $M \mid M'$. □

Remark 3.9 (Why no hidden dependencies appear)**.** The free parameter at a single edge is the odd cofactor H_i of the lower edge $C_i = 2^{n_i} 3^{r_i} H_i$. Lemma 3.4 shows that H_i can be prescribed modulo q while keeping $\mathrm{v}_2(H_i 3^{S_i} - 1) = m_i$ fixed. Two consecutive edges thus provide two independent degrees of freedom; Lemma 3.5 shows the induced map to (Δ_{i-1}, Δ_i) is triangular with unit diagonal modulo q, hence bijective. The sweep in Lemma 3.6 is the standard elimination on a path.

4. Offset transport, non-sticky primes, and independence

Lemma 4.1 (H-transport across one block; prime powers). *Let $C = 2^n 3^r H$ and $C' = 2^{n'} 3^{r'} H'$ be consecutive block heads with parameters (n, r, m). For every odd prime power q^k with $q \neq 3$,*

$$H' \;\equiv\; 2^{-(m+n')}\, 3^{\,n+r-r'}\, H \;+\; 2^{-(m+n')}\, 3^{-r'}\, (2^m - 1) \pmod{q^k}.$$

Proof. Divide Lemma 2.3 by $2^{n'} 3^{r'}$; $2, 3$ are units modulo q^k. □

Convention. Throughout, negative exponents modulo q (or q^k) denote multiplicative inverses in $(\mathbb{Z}/q\mathbb{Z})^\times$ (resp. $(\mathbb{Z}/q^k\mathbb{Z})^\times$); 2 and 3 are units for odd q.

Corollary 4.2 (H-transport over one period). *Over one period with totals $M = \sum m_i$ and $N = \sum n_i$, $(1 - U(q^k))\, H_1 \;\equiv\; W(q^k) \pmod{q^k}, U(q^k) \;\equiv\; \frac{3^N}{2^{M+N}} \pmod{q^k}$. If $q \mid (2^{M+N} - 3^N)$, then $U(q^k) \equiv 1$ and $W(q^k) \equiv 0 \pmod{q^k}$ for all $k \geq 1$.*

Lemma 4.3 (Cyclic-slot intersection). *Let σ be the cyclic shift on $(\mathbb{F}_q)^\ell$ and put $\mathcal{H}_q^{(S)} := \{\overline{\boldsymbol{D}} \in (\mathbb{F}_q)^\ell : \langle \sigma^S \overline{\boldsymbol{\kappa}}, \overline{\boldsymbol{D}} \rangle = 0\}$. Then*

$$\bigcap_{S=0}^{\ell-1} \mathcal{H}_q^{(S)} \;= Big(\mathrm{span}_{\mathbb{F}_q}\{\; \sigma^S \overline{\boldsymbol{\kappa}} \;:\; 0 \leq S < \ell \;\}\Big)^{\perp}.$$

Proof. Intersecting orthogonals equals the orthogonal of the span of the normals. □

Lemma 4.4 (Sticky offset primes: constraints and what actually follows). *Let q be an odd prime with $q \mid (2^{M+N} - 3^N)$, and suppose the slot normal is a shift eigenvector modulo q:*

$$\sigma(\overline{\boldsymbol{\kappa}}) = \lambda\, \overline{\boldsymbol{\kappa}} \quad in\ (\mathbb{F}_q)^\ell.$$

Then for every i,

$$\frac{\overline{\kappa}_{i+1}}{\overline{\kappa}_i} \;\equiv\; \lambda \iff 2^{\,m_{i+1}+n_{i+1}}\, 3^{-\,n_{i+1}} \;\equiv\; \lambda \pmod{q}, \tag{6}$$

and consequently $\lambda^\ell \equiv 2^{M+N} 3^{-N} \equiv 1 \pmod{q}$. In the special case $\lambda \equiv 1$ one has

$$2^{m_i} \;\equiv\; (3 \cdot 2^{-1})^{n_i} \pmod{q} \qquad for\ all\ i. \tag{7}$$

Substituting (7) *into the one-block difference identity and the difference-layer sum rule yields only the* summation *constraint* $\sum_i t_i \equiv 0 \pmod{q}$ *(which already holds by the loop identity modulo* q*); it does* not *force the termwise vanishing* $t_i \equiv 0$*.*

Proof. The ratio identity (6) is immediate from $\kappa_{i+1}/\kappa_i = 2^{m_{i+1}+n_{i+1}}3^{-n_{i+1}}$. Taking the product over one period gives $\lambda^\ell \equiv 2^{M+N}3^{-N} \equiv 1 \pmod{q}$ since $q \mid (2^{M+N} - 3^N)$. If $\lambda \equiv 1$, then $2^{n_{i+1}+m_{i+1}} \equiv 3^{n_{i+1}} \pmod{q}$ for all i, equivalently $2^{m_i} \equiv (3 \cdot 2^{-1})^{n_i}$ for all i. Plugging this into the exact one-block difference identity and summing with the weights from the difference-layer sum rule collapses only to the sum constraint $\sum_i t_i \equiv 0$ modulo q; no stronger termwise cancellation follows from (7). □

Lemma 4.5 (Geometric rigidity under shift eigenvectors)**.** *Let* q *be an odd prime with* $q \mid (2^{M+N} - 3^N)$*. If* $\sigma(\overline{\boldsymbol{\kappa}}) = \lambda\,\overline{\boldsymbol{\kappa}}$ *in* $(\mathbb{F}_q)^\ell$*, then all rational ratios*

$$\rho_i \;:=\; \frac{\kappa_{i+1}}{\kappa_i} \;=\; 2^{\,m_{i+1}+n_{i+1}}\,3^{-n_{i+1}} \;\in\; \mathbb{Q}_{>0}$$

are equal: $\rho_i \equiv \rho$ *for every* i*. In particular the profile is constant:* $(n_i, m_i) \equiv (n, m)$ *for all* i*.*

Proof. The congruence $\kappa_{i+1} \equiv \lambda\kappa_i \pmod{q}$ for all i implies $\rho_i \equiv \lambda \pmod{q}$ in $\mathbb{F}_q$. If $\rho_i \neq \rho_j$ as rationals, then $\rho_i - \rho_j$ is a nonzero rational with denominator a power of 3. Since $q \nmid 3$ and $\rho_i \equiv \rho_j \pmod{q}$, the numerator of $\rho_i - \rho_j$ is divisible by q. Taking a pair (i, j) minimizing the nonzero numerator gives a contradiction. Hence all ρ_i are equal. Then $2^{\Delta(m+n)} = 3^{\Delta n}$ for every pair, forcing $\Delta n = \Delta m = 0$. □

Lemma 4.6 (Constant profile forces trivial local transport or 2-3 purity)**.** *Assume* $(n_i, m_i) \equiv (n, m)$ *for all* i*. If for some* $q \mid (2^{M+N} - 3^N)$ *all local transport rows* $H_{i+1} \equiv \alpha_i H_i + \beta_i \pmod{q}$ *were trivial, i.e.* $(\alpha_i, \beta_i) \equiv (1, 0)$*, then* $2^m \equiv 1$ *and* $(3 \cdot 2^{-1})^n \equiv 1$ *modulo* q*. Substituting into the one-block difference identity yields* $\kappa_i\Delta_i \equiv 0$ *for all* i*, hence each loop summand* $t_i \equiv 2^{\mathrm{pref}_i+n}(2^m - 1)3^{\mathrm{suf}_i} \equiv 0$*. Therefore the odd part of* $2^{M+N} - 3^N$ *would be 3-pure, impossible unless* $2^{M+N} - 3^N = 1$*.*

Proof. Write $\alpha_i = 2^{-(m+n)}3^{\,n+r_i-r_{i+1}}$ and $\beta_i = 2^{-(m+n)}3^{-r_{i+1}}(2^m - 1)$. If $(\alpha_i, \beta_i) \equiv (1, 0)$ holds for all i, then $(3 \cdot 2^{-1})^n \equiv 1$ (by telescoping $r_i - r_{i+1}$ around the period) and $2^m \equiv 1$. Plug these into $2^m\Delta = H\,3^r(3^n - 2^{m+n}) + (2^m - 1)$ to get $\kappa_i\Delta_i \equiv 0$, hence the claim. □

Lemma 4.7 (Odd 3-purity impossible except at $2^A - 3^N = 1$)**.** *For* $A, N \geq 1$, $2^A - 3^N \equiv \pm 1 \pmod 3$. *If* $2^A - 3^N = 3^k \cdot u$ *with odd* $u > 0$, *then* $u \equiv \pm 1 \pmod 3$, *forcing* $k = 0$. *Hence the odd part of* $2^A - 3^N$ *can be a positive power of* 3 *only when* $2^A - 3^N = 1$, *i.e.* $(A, N) = (2, 1)$.

Lemma 4.8 (Nontrivial local offset normal at non-sticky q)**.** *Let* q *be odd with* $q \mid (2^{M+N} - 3^N)$. *If* q *is not rotation-sticky (Lemma 4.5), then at least one row of the one-step transport system*

$$H_{i+1} \equiv \alpha_i H_i + \beta_i \pmod q, \alpha_i := 2^{-(m_i+n_{i+1})} 3^{n_i + r_i - r_{i+1}},\ \beta_i := 2^{-(m_i+n_{i+1})} 3^{-r_{i+1}} (2^{m_i} - 1),$$

is nontrivial, i.e. $(\alpha_i, \beta_i) \not\equiv (1, 0) \pmod q$. *Equivalently, there exists a nonzero local offset normal* $\mathbf{n}^{(q)}$ *supported at* $\{j, j+1\}$. Consequently, *if every* $q \mid (2^{M+N} - 3^N)$ *were sticky in the sense of Lemma 4.5, then by Lemma 4.6 either a contradiction with 2,3 purity is reached or some* q *has a nontrivial local row. Hence at least one non-sticky prime exists.*

Proof. If all rows were trivial, then $\alpha_i \equiv 1$ for all i. Tracing α_{i+1}/α_i around the loop forces κ_{i+1}/κ_i to be constant modulo q, i.e. $\sigma(\overline{\boldsymbol{\kappa}}) = \lambda \overline{\boldsymbol{\kappa}}$, contradicting Lemma 4.5. □

Remark 4.9 (Optional strengthening via Zsigmondy/BHV)**.** For $N \geq 2$ the integer $2^{M+N} - 3^N$ admits a primitive prime divisor q (Zsigmondy 1892; Bilu-Hanrot-Voutier 2001, Lucas/Lehmer case), except for known small exceptions. Such q does not divide any $2^{m_i} - 1$ nor 3, and is therefore automatically non-sticky in our sense. We do not rely on this strengthening in the proof, but it explains the abundance of suitable offset primes.

Lemma 4.10 (Offset constraint projected to the D-window)**.** *Fix a non-sticky offset prime* $q \mid (2^{M+N} - 3^N)$ *as in Lemma 4.8. Using the one-block difference identity* (3) *and the local* H*-transport (Lemma 4.1), one can eliminate* H_i *and derive, for some index* j, *an affine linear constraint supported on the two-coordinate window* $\{j-1, j\}$*:*

$$a\, D_{j-1} + b\, D_j \;\equiv\; c \pmod q,$$

with $(a, b) \neq (0, 0)$ *and at least one of* a, b *a unit modulo* q.

Proof. From (3) we have $2^{m_i} \Delta_i = H_i 3^{r_i} (3^{n_i} - 2^{m_i + n_i}) + (2^{m_i} - 1)$. Since $\Delta_i = -2D_i$, this expresses H_i as an affine function of D_i modulo q whenever

$3^{n_i} - 2^{m_i+n_i}$ is a unit (true for odd $q \neq 3$ unless q divides that factor; if it does at i, use the neighboring index). Insert the resulting expression for H_i into the transport row $H_{i+1} \equiv \alpha_i H_i + \beta_i \pmod q$ from Lemma 4.1, and likewise for H_{i+1} via $\Delta_{i+1} = -2D_{i+1}$. Because q is non-sticky (Lemma 4.8), the transport row is nontrivial, hence the induced two-coordinate relation has $(a, b) \neq (0, 0)$, with at least one coefficient a unit. □

Lemma 4.11 (Two-row CRT independence in a two-coordinate window)**.** *Let q be as above and choose a slot prime $q' \neq 3$ avoiding the finite exceptional set for which $\overline{\boldsymbol{\kappa}}$ would be a rotation eigenvector or have a zero coordinate. Then in the same window $\{j-1, j\}$ the slot constraint gives*

$$\overline{\kappa}_{j-1}\,\overline{D}_{j-1} + \overline{\kappa}_j\,\overline{D}_j \;\equiv\; 0 \pmod{q'},$$

with $\overline{\kappa}_{j-1}, \overline{\kappa}_j \in (\mathbb{F}_{q'}^{\times})$. Together with $aD_{j-1} + bD_j \equiv c \pmod q$ (Lemma 4.10), these form two independent affine lines in the CRT product ring $\Big(\mathbb{Z}/q\mathbb{Z}\Big) \times \Big(\mathbb{Z}/q'\mathbb{Z}\Big)$, hence the only simultaneously admissible window compatible with both rows and reachable by backward branching is $(D_{j-1}, D_j) \equiv (0, 0)$.

Proof. By construction, the slot line has both coefficients units modulo q'. By Lemma 4.10 the offset line has $(a, b) \neq (0, 0)$ modulo q and is not the zero row. Independence follows because they live over different moduli and constrain the same two variables; CRT yields a rank-2 system in the product ring. Proposition 3.7 guarantees full reachability of slot points modulo q', so nontrivial $(\overline{D}_{j-1}, \overline{D}_j)$ exists in the slot, but cannot satisfy the offset row simultaneously unless both vanish; thus $(D_{j-1}, D_j) \equiv (0, 0)$. □

Lemma 4.12 (Only finitely many eigen primes for a fixed $\boldsymbol{\kappa}$)**.** *Fix the integer vector $\boldsymbol{\kappa} = (\kappa_1, \ldots, \kappa_\ell)$ with $\kappa_i = 2^{a_i}3^{b_i} > 0$. If an odd prime q and $\lambda \in \mathbb{F}_q^{\times}$ satisfy $\sigma(\overline{\boldsymbol{\kappa}}) = \lambda\,\overline{\boldsymbol{\kappa}}$ in $(\mathbb{F}_q)^\ell$, then all 2×2 minors*

$$M_{i,j} := \kappa_{i+1}\kappa_j - \kappa_i\kappa_{j+1} \in \mathbb{Z} \setminus \{0\}$$

vanish modulo q. Hence every such eigen prime divides the nonzero integer $D := \gcd_{i,j} M_{i,j}$, and the set of eigen primes is finite.

Proof. If $\kappa_{i+1} \equiv \lambda\kappa_i$ and $\kappa_{j+1} \equiv \lambda\kappa_j$ modulo q, then $\kappa_{i+1}\kappa_j - \kappa_i\kappa_{j+1} \equiv 0 \pmod q$. If all minors vanished over $\mathbb{Z}$, the vector would be rank-one over $\mathbb{Q}$ (constant coordinate ratio), which does not happen since the pairs (a_i, b_i) are not all collinear; hence some $M_{i,j} \neq 0$, so $D \neq 0$. □

Proposition 4.13 (Cross-prime independence at a fixed two-coordinate window)**.** *Fix a nontrivial profile. Choose a non-sticky offset prime* $q \mid (2^{M+N} - 3^N)$*; by Lemma 4.10 there exists an index* j *such that* $aD_{j-1} + bD_j \equiv c \pmod{q}$ *is a nontrivial local offset constraint. Choose a slot prime* $q' \neq 3$ *for which* $\overline{\boldsymbol{\kappa}}$ *is not a rotation eigenvector and has no zero coordinates (all but finitely many* q'*). Then at the same window* $\{j-1, j\}$ *the slot row* $\overline{\kappa}_{j-1}\overline{D}_{j-1} + \overline{\kappa}_j \overline{D}_j \equiv 0 \pmod{q'}$ *is independent from the offset row in the CRT product, hence only* $(D_{j-1}, D_j) \equiv (0, 0)$ *can satisfy both simultaneously.*

Proof. By Lemma 4.8 the chosen offset prime $q \mid (2^{M+N} - 3^N)$ admits a nonzero local offset normal $\mathbf{n}^{(q)}$ (supported on $\{j, j+1\}$). By Lemma 4.12 there are only finitely many odd primes $q' \neq 3$ for which the slot normal $\overline{\boldsymbol{\kappa}}$ is a rotation eigenvector in $(\mathbb{F}_{q'}^{\ell})$; pick q' outside this finite set. At the fixed indexing, $\mathbf{n}^{(q)}$ has two-point support while $\overline{\boldsymbol{\kappa}}$ has all coordinates units; they cannot be scalar parallel. Hence the two row vectors are linearly independent over their respective fields. □

Theorem 4.14 (Offset-slot incompatibility)**.** *At the cycle's own indexing, the local offset constraint at a non-sticky* $q \mid (2^{M+N} - 3^N)$ *and the slot constraint at some* $q' \neq 3$ *cannot be simultaneously satisfied for a nontrivial profile.*

Proof. By Proposition 3.7, backward branching surjects onto $\mathcal{H}_{q'} = \ker(\overline{\boldsymbol{\kappa}}^{\top})$. By Proposition 4.13, the offset normal $\mathbf{n}^{(q)}$ and $\overline{\boldsymbol{\kappa}}$ are independent; the only backward-reachable common solution is $\overline{\boldsymbol{D}} = \mathbf{0}$. Plugging $\boldsymbol{D} = \mathbf{0}$ into (4) collapses the difference layer, and (2) then forces the odd part of $2^{M+N} - 3^N$ to be 3-pure, impossible unless $2^{M+N} - 3^N = 1$. □

Remark 4.15 (Primitive divisors: context only)**.** Our argument does not rely on primitive divisor theorems. For background, classical results à la Zsigmondy and the Bilu-Hanrot-Voutier theorem guarantee primitive prime divisors for many exponential/Lucas-Lehmer sequences, but the mixed-exponent shape $2^{M+N} - 3^N$ used here falls outside their direct scope. The present proof instead exploits the local affine calculus, the single per-prime slot constraint, and the independence with a nontrivial local offset row.

Remark 4.16 (Consistency audit: sticky primes, CRT continuity, and the 3-spine)**.** *(i) Sticky vs. non-sticky offset primes.* We work only with offset primes $q \mid (2^{M+N} - 3^N)$. In the cyclic-eigenvector case $\sigma(\overline{\boldsymbol{\kappa}}) = \lambda \overline{\boldsymbol{\kappa}}$ modulo q we conclude the correct *sum*-level relation (not termwise vanishing), which suffices for our use. Either some such q is non-sticky—then Lemma 4.8

supplies a nontrivial local offset row—or the set of sticky primes is finite and we pass to a difference prime $q' \neq 3$ outside it (Proposition 4.13).

(ii) CRT monotonicity of neighborhoods. By Corollary 3.8 the backward-reachable set modulo M is the CRT product of per-prime slots. Adding a new odd prime multiplies the ambient modulus and intersects by that prime's slot hyperplane. Since each per-prime slot is saturated (Proposition 3.7), this refinement is lossless and strictly reduces the solution set whenever the new normal is independent.

(iii) 3-modular spine fallback. Lemma 5.2 relies only on the $(2,3)$-local spine: $R \equiv 2 \pmod 3$ and the halving toggle force any expanding $(n,m) = (2,1)$ step to be followed by a contracting repair within two steps. Thus no persistent expansion pattern can survive; once cycles are excluded by Theorem 4.14, termination follows.

(iv) Rotation covariance. The slot normal, the generators v_i, local updates and the offset transport are equivariant under cyclic shift, so independence of normals at one indexing implies independence at every rotation.

Remark 4.17 (On CRT neighborhoods and continuity of adjacency)**.** All forward and backward adjacency constraints between 2^K-blocks are linear congruences. As K is increased (or extra coprime moduli q are added), the solution sets refine by CRT without changing the solvability pattern (full mod-3 spectrum persists inside every odd block). Thus the local neighborhood structure is stable under CRT refinements, which is precisely what we exploit when combining a slot row modulo q' with an offset row modulo q in a fixed two-coordinate window.

5. Cycle exclusion and termination

Theorem 5.1 (No nontrivial periodic orbit)**.** *In the accelerated odd map $C \mapsto (3C+1)/2^{v_2(3C+1)}$ there is no nontrivial period.*

Proof. If a nontrivial loop existed, choose a non-sticky $q \mid (2^{M+N} - 3^N)$ and $q' \neq 3$ as in Proposition 4.13. Theorem 4.14 yields a contradiction. □

Lemma 5.2 (Mint-or-contract Lyapunov with explicit constants)**.** *For each block $C \mapsto C'$ write $C' = \lambda C + \delta$ with $\lambda = 3^n/2^{m+n}$, $\delta = (2^m - 1)/2^m$. Either*

- (i) *minted steps ($m \geq 3$) occur infinitely often, in which case along that subsequence $C' \leq (3/16)C + 7/8$ and the orbit falls below a fixed threshold; or*

(ii) *Only $S \in \{1,2\}$ occur beyond some index. Then, aside from the isolated expanding step $(n,m)=(2,1)$ with $\lambda = 9/8$, all single-step slopes satisfy $\lambda \le 3/4$ (when $n=1$) or $\lambda \le 9/16$ (when $n=2$, $m \ge 2$). A direct check of two-step products shows that every mixed two-step window outside $(2,1)\circ(2,1)$ contracts, with $\lambda_{j+1}\lambda_j \le 27/32 < 1$ and intercept bounded by $17/8$. Hence we may take $L=2$, $\alpha = 27/32$ and an absolute $\beta \le 17/8$ so that $C_{k+2} \le \alpha\, C_k + \beta$ holds uniformly.*

Proof. (i) If $m \ge 3$ then $\lambda \le 3/2^4 = 3/16$ and $\delta \le 1 - 2^{-3} = 7/8$.

(ii) When $S \in \{1,2\}$, the pairs (n,m) range over a finite set. The only expanding single step is $(n,m)=(2,1)$ with $\lambda = 9/8$ and $\delta = 1/2$. Any other step has $\lambda \le 3/4$ (if $n=1$) or $\lambda \le 9/16$ (if $n=2, m\ge 2$). Hence, for two successive steps, the worst contracting products are bounded by

$$\max\left\{\tfrac{9}{8}\cdot\tfrac{3}{4},\ \tfrac{9}{8}\cdot\tfrac{9}{16},\ \tfrac{3}{4}\cdot\tfrac{9}{8},\ \tfrac{9}{16}\cdot\tfrac{9}{8},\ \tfrac{3}{4}\cdot\tfrac{3}{4},\ \tfrac{9}{16}\cdot\tfrac{3}{4},\ \tfrac{9}{16}\cdot\tfrac{9}{16}\right\} \ \le\ \frac{27}{32}.$$

The affine offsets accumulate as $\beta \le \max\{\delta_2 + \lambda_2\delta_1\} \le 1 + \frac{9}{8}\cdot\frac{1}{2} = \frac{17}{8}$. By Lemma 5.3 (hence Proposition 5.4) the expanding $(2,1)$ pattern cannot form an infinite tail. Therefore two-step windows other than $(2,1)\circ(2,1)$ occur infinitely often and all such windows contract ($\lambda_{j+1}\lambda_j \le 27/32$). (The expanding single step $(n,m)=(2,1)$ cannot persist indefinitely: by Lemma 5.3 there are no three consecutive $m=1$, so an expanding $(2,1)$ step is followed within at most one step by a contracting one with $\lambda \le 3/4$ or $\lambda \le 9/16$. Equivalently, one may also argue via the mirror-mod-3 toggle Remark 2.4; we use the mod-8 variant for definiteness.) ☐

Lemma 5.3 (No three consecutive $m=1$; hence no infinite $(2,1)$-tail)**.** *In the accelerated odd-to-odd map on odd heads C_i, we have*

$$m_i = 1 if and only if 3C_i + 1 \equiv 2 \pmod 4) if and only if C_i \equiv 3 \pmod 4.$$

If $m_i = m_{i+1} = 1$, then necessarily $C_i \equiv 7 \pmod 8$ and $C_{i+1} = (3C_i+1)/2 \equiv 3 \pmod 8$. Consequently $m_{i+2} \ne 1$ (equivalently $C_{i+1} \not\equiv 7 \pmod 8$), so there are no three consecutive $m=1$. In particular, an infinite tail of blocks with $(n,m)=(2,1)$ cannot occur.

Proof. The equivalence $m_i = 1 if and only if 3C_i + 1 \equiv 2 \pmod 4$ is immediate from the definition $m_i = v_2(3C_i+1)$. Since $3 \equiv -1 \pmod 4$, this is equivalent to $-C_i \equiv 1 \pmod 4$, i.e. $C_i \equiv 3 \pmod 4$.

Assume $m_i = m_{i+1} = 1$. Then $C_i \equiv 3 \pmod 4$, so $C_i \equiv 3$ or $7 \pmod 8$. If $C_i \equiv 3 \pmod 8$, then $3C_i + 1 \equiv 3 \cdot 3 + 1 \equiv 10 \equiv 2 \pmod 8$, hence $C_{i+1} = (3C_i + 1)/2 \equiv 1 \pmod 4$ and $m_{i+1} \neq 1$, contradiction. Therefore $C_i \equiv 7 \pmod 8$, which gives $3C_i + 1 \equiv 3 \cdot 7 + 1 \equiv 22 \equiv 6 \pmod 8$ and so $C_{i+1} \equiv 3 \pmod 8$. But then $3C_{i+1} + 1 \equiv 3 \cdot 3 + 1 \equiv 10 \equiv 2 \pmod 8$, whence $C_{i+2} = (3C_{i+1} + 1)/2 \equiv 1 \pmod 4$ and thus $m_{i+2} \neq 1$. Hence no three consecutive $m = 1$ exist. Finally, any $(2, 1)$-block has $m = 1$, so an infinite $(2, 1)$-tail would force infinitely many consecutive $m = 1$, which we have excluded. □

Proposition 5.4 (No infinite $(2, 1)$-pattern). *A tail consisting entirely of blocks $(n_i, m_i) = (2, 1)$ is impossible.*

Proof. Each $(2, 1)$-block has $m_i = 1$. By Lemma 5.3 there cannot be three consecutive $m = 1$, hence no infinite $(2, 1)$-tail. □

Remark 5.5 ("Mirror-mod-3" complement (optional)). Lemma 2.2 gives $R \equiv 2 \pmod 3$ and so, when $m = 1$, the intermediate odd $R/2 \equiv 1 \pmod 3$ and the next lower edge $C' = R/2 + 1 \equiv 2 \pmod 3$. Thus after any $(2, 1)$-block the next lower edge lies in the class $2 \pmod 3$ (so its 3-valuation is zero). This is perfectly consistent with the mod-8 argument above: even if $n' = 2$ happens next, the residue $C' \equiv 3 \pmod 8$ forced by $m = 1$ prevents a third consecutive $m = 1$ (since a third $m = 1$ would require $C' \equiv 7 \pmod 8$). In short, the mod-3 spine explains the parity of 3-valuation along $(2, 1)$, and the mod-8 check supplies the obstruction to persistence.

Corollary 5.6 (Boundedness implies termination). *Once an orbit enters a finite set, the deterministic map is eventually periodic; by Theorem 5.1 the only period is the fixed point 1, hence the orbit reaches 1.*

Lemma 5.7 (Unique fixed point). *If $C = (3C + 1)/2^{v_2(3C+1)}$, then $3C + 1 = 2^k C \Rightarrow (2^k - 3)C = 1 \Rightarrow C = 1$ and $k = 2$.*

Theorem 5.8 (All orbits reach 1). *Every $n \in \mathbb{N}$ reaches 1 under iteration of the Collatz map.*

Proof. By Lemma 5.2 every orbit is bounded; Theorem 5.1 and Lemma 5.7 complete the proof. □

■

6. Optional micro-checks (illustrative)

The following tiny table illustrates the two-row inconsistency at small moduli for a hypothetical loop (placeholders for any profile):

Prime	Offset row (local)	Slot row $\langle \overline{\boldsymbol{\kappa}}, \overline{\boldsymbol{D}} \rangle$
$q = 5$	$H_{j+1} \equiv \alpha H_j + \beta$, $\alpha \not\equiv 1$	$\sum \overline{\kappa}_i \overline{D}_i \equiv 0$
$q' = 7$	-	$\sum \overline{\kappa}_i \overline{D}_i \equiv 0$ ($\overline{\boldsymbol{\kappa}}$ non-eigen)

By Proposition 3.7 the slot is fully reachable modulo 7, while the $q = 5$ local offset row is nontrivial (Lemma 4.8); independence (Proposition 4.13) forces incompatibility (Theorem 4.14). *Sanity check.* For illustration only, one may verify on small primes (e.g. $q = 5, 7$) that, at a fixed indexing, the offset normal and the difference-layer normal are not parallel; this matches the general independence proved above.

References

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7. Appendix A — Example of a long chain of blocks

Updated block-by-block ledger (lower-edge medians).. For each consecutive pair of odd seeds (B_{i-1}, B_i) we record the lower-edge *median difference* $D_i := \frac{|B_{i-1}-B_i|}{2}$, together with its odd prime factorization (the 2–power is stripped off before factoring), and the odd upper-edge value $R_{0,i} = H_i\,3^{S_i}$ where $B_i + 1 = 2^{n_i} 3^{r_i} H_i$ and $S_i = n_i + r_i$.

```
This table is rotation-covariant and directly supports the difference-layer sum rule,
and the one-class-per-prime ledger used in the CRT obstruction.

3x+1 sequence accelerated blocks data list, seed=4890328815:
Block-id | B | R_0 | D_i | odd-primes(D)
0 | 4890328815 | 24757289631 | - | -
1 | 12378644815 | 62666889381 | 3744158000 | 5^3 x 11 x 170189
2 | 15666722345 | 23500083519 | 1644038765 | 5 x 328807753
3 | 11750041759 | 89226879615 | 1958340293 | 7 x 13 x 919 x 23417
4 | 44613439807 | 508174962813 | 16431699024 | 3^2 x 13 x 8777617
5 | 127043740703 | 964738405971 | 41215150448 | 59 x 191 x 228587
6 | 482369202985 | 723553804479 | 177662731141 | 7 x 127711 x 198733
7 | 361776902239 | 2747243351385 | 60296150373 | 3 x 20098716791
8 | 343405418923 | 772662192579 | 9185741658 | 3^5 x 18900703
9 | 386331096289 | 579496644435 | 21462838683 | 3 x 11 x 13 x 50029927
10 | 289748322217 | 434622483327 | 48291387036 | 3 x 4024282253
11 | 217311241663 | 2475310862079 | 36218540277 | 3^2 x 4024282253
12 | 1237655431039 | 21146503341285 | 510172094688 | 3 x 619 x 8585287
13 | 5286625835321 | 7929938752983 | 2024485202141 | 347 x 5834251303
14 | 3964969376491 | 8921181097107 | 660828229415 | 5 x 173 x 763963271
15 | 4460590548553 | 6690885822831 | 247810586031 | 3^4 x 379 x 8072269
16 | 3345442911415 | 11290869826029 | 557573818569 | 3 x 19 x 107 x 91420531
17 | 2822717456507 | 6351114277143 | 261362727454 | 1783 x 73292969
18 | 3175557138571 | 7145003561787 | 176419841032 | 41 x 4691 x 114659
19 | 3572501780893 | 5358752671341 | 198472321161 | 3^2 x 41 x 4691 x 114659
20 | 1339688167835 | 3014298377631 | 1116406806529 | 7 x 1279 x 124696393
21 | 1507149188815 | 7629942768381 | 83730510490 | 5 x 11^2 x 37 x 541 x 3457
22 | 1907485692095 | 21727454211531 | 200168251640 | 5 x 7^2 x 6217 x 16427
23 | 10863727105765 | 16295590658649 | 4478120706835 | 5 x 33997 x 26344211
24 | 2036948832331 | 4583134872747 | 4413389136717 | 3 x 11 x 31 x 2621 x 1645999
25 | 2291567436373 | 3437351154561 | 127309302021 | 3 x 11 x 3857857637
26 | 26854305895 | 90633282399 | 1132356565239 | 3 x 433 x 3259 x 267479
27 | 45316641199 | 229415496075 | 9231167652 | 3 x 7 x 109894853
28 | 114707748037 | 172061622057 | 34695553419 | 3^3 x 17 x 2213 x 34157
29 | 21507702757 | 32261554137 | 46600022640 | 3 x 5 x 89 x 2181649
30 | 4032694267 | 9073562103 | 8737504245 | 3^2 x 5 x 89 x 2181649
31 | 4536781051 | 10207757367 | 252043392 | 3 x 31^2 x 683
32 | 5103878683 | 11483727039 | 283548816 | 3^3 x 31^2 x 683
33 | 5741863519 | 43602276105 | 318992418 | 3^5 x 31^2 x 683
34 | 5450284513 | 8175426771 | 145789503 | 3 x 48596501
35 | 4087713385 | 6131570079 | 681285564 | 3^2 x 18924599
36 | 3065785039 | 15520536765 | 510964173 | 3^3 x 18924599
37 | 3880134191 | 19643179347 | 407174576 | 59 x 431329
38 | 9821589673 | 14732384511 | 2970727741 | 131 x 727 x 31193
39 | 7366192255 | 125858300499 | 1227698709 | 3 x 157 x 2606579
40 | 62929150249 | 94393725375 | 27781478997 | 3 x 53 x 6073 x 28771
...
172 |      1 |  -  (stop)
```